Treatise on Laplace Transforms

Johar M. Ashfaque
09047995
University Of Kent

April 5, 2011

1 Introduction

We present in this article, a study of the Laplace transforms. The theory of the Laplace transform, as [1, 3] suggests, has a long and rich history. Many mathematicians can be named, among which Euler, Lagrange and Laplace played important roles, as [3] points out, in realising the importance of the Laplace transform to solve not only differential equations but also difference equations. Euler, as highlighted in [1], used the Laplace transform in order to solve certain differential equations, whereas it was Laplace who understood the true essence of the theory of the Laplace transform in solving both differential and difference equations. For further reference, [1, 3] are recommended.

2 Laplace Transform

For a complex-valued function x, defined for $t > 0$, the Laplace transform of $x(t)$ is defined by

$$X(s) = \int_0^\infty x(t)e^{-st}dt \tag{1}$$

for all $s \in \mathbb{R}$. Alternatively, we may use the expression

$$\mathcal{L}\{x(t)\} = X(s)$$

to denote the Laplace transform.

Example 1: Let $x(t) = e^{-at}$ for $a \in \mathbb{R}$. Then by direct integration

$$X(s) = \int_0^\infty e^{-(s+a)}dt = \frac{1}{s+a} \quad , \quad s > a \quad .$$

In a similar fashion we can obtain the Laplace transform for e^{at} which can be seen in [3, 5].

Example 2: Let $x(t) = \sin(\omega t)$, for $\omega \in \mathbb{R}$. Then

$$X(s) = \int_0^\infty \sin(\omega t)\, e^{-st}dt \quad .$$

Integrating the right hand side by parts twice, we obtain

$$\int_0^\infty \sin(\omega t)\, e^{-st}dt = \frac{w}{s^2} - \frac{\omega^2}{s^2}\int_0^\infty \sin(\omega t)e^{-st}dt \quad ,$$

Rearranging we find

$$\left(\frac{s^2+\omega^2}{s^2}\right)\int_0^\infty \sin(\omega t)\, e^{-st}dt = \frac{w}{s^2} \Longrightarrow \int_0^\infty \sin(\omega t)\, e^{-st}dt = \frac{w}{s^2+\omega^2} \quad .$$

Thus

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

as required.
A different approach to evaluating the Laplace transform for $x(t) = \sin(\omega t)$ can be found in [3].

Tables of the Laplace transforms of various functions can be found in many books and formulae sheets. However, I highly recommend [2, 3, 4].

3 Properties of the Laplace Transform

In this section, we look at the standard properties of the Laplace transform.

3.1 Linearity of the Laplace Transform

Linearity of the Laplace transform, as [3, 7] highlight, is an important result which states:

$$\mathcal{L}\{Cx(t)\} = C\mathcal{L}\{x(t)\} = CX(s) \quad , \quad C \in \mathbb{R} \quad ; \tag{2}$$

$$\mathcal{L}\{Ax(t) + By(t)\} = AX(s) + BY(s) \quad , \quad A, B \in \mathbb{R} \quad . \tag{3}$$

3.2 Derivatives of the Laplace Transform

3.2.1 First Derivative

The first derivative of the Laplace transform is given by

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0) \quad . \tag{4}$$

Proof. Directly from the definition of the Laplace transform, we have

$$\mathcal{L}\{x'(t)\} = \int_0^\infty e^{-st} x'(t) dt$$

The integral on the right hand side, can be integrated by parts once to obtain

$$\int_0^\infty e^{-st} x'(t) dt = x(t) e^{-st} \Big|_0^\infty + s \int_0^\infty x(t) e^{-st} dt = -x(0) + s \int_0^\infty e^{-st} x(t) dt \quad .$$

Hence

$$\mathcal{L}\{x'(t)\} = -x(0) + sX(s) = sX(s) - x(0)$$

as required. □

3.2.2 The n^{th} Derivative

The equation of the n^{th} derivative of the Laplace transform is given by

$$f^{(n)}(t) = s^n \mathcal{L}\{x()t\} - s^{n-1} x(0) - \ldots - x^{n-1}(0) \quad . \tag{5}$$

This equation can be found in the tables of the Laplace transform in [2, 3].

3.3 Shift property of the Laplace Transform

The shift property of the Laplace transform states

$$\mathcal{L}\{x(t) e^{at}\} = X(s-a) \quad . \tag{6}$$

Proof. By direct integration

$$\mathcal{L}\{x(t) e^{at}\} = \int_0^\infty x(t) e^{-(s-a)t} dt = X(s-a) \quad .$$

□

In many books, this property is referred to as the "First Shift Theorem".

3.4 Scaling property of the Laplace Transform

The scaling property of the Laplace transform states

$$\mathcal{L}\{x(at)\} = \frac{1}{a} X\left(\frac{s}{a}\right) \quad . \tag{7}$$

Proof. By the definition of the Laplace transform

$$\mathcal{L}\{x(at)\} = \int_0^\infty e^{-st} x(at) dt = \frac{1}{a} \int_0^\infty e^{-(\frac{s}{a})y} x(y) dy = \frac{1}{a} X\left(\frac{s}{a}\right)$$

as required. □

3.5 Laplace Transform of t^n

The Laplace transform of t^n, can be found in [3, 4, 5, 7], is defined to be

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad . \tag{8}$$

The systematic proof can be found in [4].

Example: The Laplace transform of t^6 is computed as

$$\mathcal{L}\{t^6\} = \frac{6!}{s^7} \quad .$$

3.6 Examples

Example 1: Let $x(t) = e^{6t} \sin(3t)$. Then applying the shift property of the Laplace transform, we have

$$\mathcal{L}\{e^{6t} \sin(3t)\} = \frac{3}{(s-6)^2 + 9}$$

as the Laplace transform for $x(t) = e^{6t} \sin(3t)$.

In general, by the shift property of the Laplace transform, we conclude that

$$\mathcal{L}\{e^{at} \sin(\omega t)\} = \frac{\omega}{(s-a)^2 + \omega^2} \quad .$$

Example 2: Let $x(t) = \cos(\omega t)$ for $\omega \in \mathbb{R}$.

Recall:

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2} \quad .$$

Then by the linearity property of the Laplace transform, we have

$$\mathcal{L}\{\cos(\omega t)\} = \mathcal{L}\left\{\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right\} = \frac{1}{2}(\mathcal{L}\{e^{i\omega t}\} + \mathcal{L}\{e^{-i\omega t}\}) \quad .$$

We can now use the shift property of the Laplace transform because we know that for $x(t) = e^{-at}$, we have $X(s) = \frac{1}{s+a}$. As a result, we obtain

$$\frac{1}{2}(\mathcal{L}\{e^{i\omega t}\} + \mathcal{L}\{e^{-i\omega t}\}) = \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right) = \frac{1}{2}\left(\frac{2s}{(s-i\omega)(s+i\omega)}\right) = \frac{1}{2}\left(\frac{2s}{(s^2+\omega^2)}\right) = \frac{s}{s^2+\omega^2} \quad .$$

Example 3: Let $x(t) = e^{9t} \cos(8t)$. Then applying the shift property of the Laplace transform, we have

$$\mathcal{L}\{e^{9t} \cos(8t)\} = \frac{s-9}{(s-9)^2 + 64}$$

as the Laplace transform for $x(t) = e^{9t} \cos(8t)$.

In general, by the shift property of the Laplace transform, we conclude that

$$\mathcal{L}\{e^{at} \cos(\omega t)\} = \frac{s-a}{(s-a)^2 + \omega^2} \quad .$$

3.7 Inverse Laplace Transform

If for a given function $X(s)$, we can find a function $x(t)$ such that $\mathcal{L}\{x(t)\} =$ X(s), then the inverse Laplace transform is denoted

$$\mathcal{L}^{-1}\{X(s)\} = x(t)$$

and is unique.

The inverse Laplace transform is linear:

$$\mathcal{L}^{-1}\{CX(s)\} = Cx(t) \quad , \quad C \in \mathbb{R} \quad ; \tag{9}$$

$$\mathcal{L}^{-1}\{AX(s) + BY(s)\} = Ax(t) + By(t) \quad , \quad A, B \in \mathbb{R} \quad . \tag{10}$$

Example: Find

$$\mathcal{L}^{-1}\left\{\frac{12s-36}{(s+5)(s-3)(s+7)}\right\} \quad .$$

We use equation (6) and equation (10), to obtain

$$\mathcal{L}^{-1}\left\{\frac{12s-36}{(s+5)(s-3)(s+7)}\right\} = \mathcal{L}^{-1}\left\{\frac{33}{8(s+5)} + \frac{3}{8(s-3)} - \frac{9}{2(s+7)}\right\} = \frac{33}{8}e^{-5t} + \frac{3}{8}e^{3t} - \frac{9}{2}e^{-7t} \quad .$$

4 Conditions for the existence of the Laplace Transform

Theorem 1. *Let $x(t)$ be a piecewise continuous function in the interval $[0,\infty)$ and is of exponential order a. Also, let*

$$|x(t)| \le Ke^{at}, \ \ t \ge 0$$

with real constants K and a, where K is positive. Then the Laplace transform $\mathcal{L}\{x(t)\} = X(s)$ exists for $s > a$.

For proof and further reference, [1, 3, 7] are recommended.

Example: $x(t) = e^{7t}\cos 5t$ is said to be of exponential order $a = 7$ since

$$|e^{7t}\cos 5t| \le e^{7t}$$

Hence for $K = 1$, the Laplace transform of $x(t)$ exists.

5 Convolution Theorem

In this section, we seek to compute the Laplace transform of a convolution. Let us begin by reminding ourselves what we mean by a convolution. The convolution, in the interval $[0,\infty)$, is defined as

$$(x*y)(t) = \int_0^t x(\tau)y(t-\tau)d\tau \quad . \tag{11}$$

Theorem 2. *The Laplace transform of two functions under convolution is*

$$\mathcal{L}\{(x*y)(t)\} = \mathcal{L}\{x(t)\}\cdot\mathcal{L}\{y(t)\} = X(s)Y(s) \quad .$$

Proof. From the definition of the Laplace transform, we have

$$\mathcal{L}\left(\int_0^t x(\tau)y(t-\tau)d\tau\right) = \int_0^\infty \left(\int_0^t x(\tau)y(t-\tau)d\tau\right)e^{-st}dt \quad .$$

To simplify the repeated integral we introduce the shifted unit step function $U(t-\tau)$, see appendix A, to obtain

$$\int_0^\infty \left(\int_0^t x(\tau)y(t-\tau)d\tau\right)e^{-st}dt = \int_0^\infty \left(\int_0^\infty U(t-\tau)x(\tau)y(t-\tau)d\tau\right)e^{-st}dt \quad .$$

Changing the order of integration, see [3, p. 187], we have

$$\int_0^\infty \left(\int_0^\infty U(t-\tau)x(\tau)y(t-\tau)d\tau\right)e^{-st}dt = \int_0^\infty x(\tau)\left(\int_0^\infty U(t-\tau)y(t-\tau)e^{-st}dt\right)d\tau \quad .$$

Using the second shift theorem, equation (13), we obtain

$$\int_0^\infty U(t-\tau)y(t-\tau)e^{-st}dt = e^{-s\tau}Y(s) \quad .$$

Therefore

$$\int_0^\infty x(\tau)\left(\int_0^\infty U(t-\tau)y(t-\tau)e^{-st}dt\right)d\tau = \int_0^\infty x(\tau)e^{-s\tau}Y(s)d\tau = Y(s)\int_0^\infty e^{-s\tau}x(\tau)d\tau = X(s)Y(s) \quad .$$

□

Example: Solve

$$f(t) + \int_0^t (t-u)f(u)\,du = \sin(2t) \quad .$$

We begin by taking the Laplace transform of both sides of the equation to obtain

$$\mathcal{L}\{f(t)\} + \mathcal{L}\{t * f(t)\} = \mathcal{L}\{\sin(2t)\} \quad .$$

Thus, by theorem (2), we have

$$\mathcal{L}\{f(t)\} + \frac{1}{s^2} \cdot \mathcal{L}\{f(t)\} = \frac{2}{s^2+4} \quad .$$

Solving for $\mathcal{L}\{f(t)\}$, gives

$$\mathcal{L}\{f(t)\} = \frac{2s^2}{(s^2+1)(s^2+4)} = -\frac{2}{3(s^2+1)} + \frac{8}{3(s^2+4)} \quad .$$

Taking the inverse Laplace transform, we obtain

$$f(t) = \frac{4}{3}\sin(2t) - \frac{2}{3}\sin(t) \quad .$$

6 Ordinary Differential Equations

One application of the Laplace transform is to solve differential equations. In this section, we consider ordinary differential equations or ODEs. The schema behind the use of Laplace transforms to solve ODEs is shown in the following diagram:

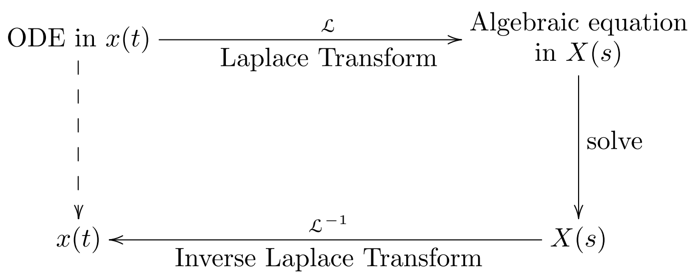

Figure 1: Ordinary differential equations(ODEs) can be solved generally, or by way of Laplace transforms, to obtain the same solution in both cases. This figure appears in [5].

Example: Find the solution to the initial value problem

$$y''(t) - y(t) = t - 2$$

given $y(2) = 3$ and $y'(2) = 0$.

We begin by moving the initial conditions to $t = 0$. This is done by setting $x(t) = y(t+2)$. Then $x'(t) = y'(t+2)$ and $x''(t) = y''(t+2)$. Substituting $t = t+2$ and $x(t) = y(t+2)$ into the differential equation, the initial value problem becomes

$$x''(t) - x'(t) = t \quad , \quad x(0) = 3 \quad , \quad x'(0) = 0 \quad .$$

Taking the Laplace transform of both sides and using the linearity of the Laplace transform, see subsection [3.1], we have

$$\mathcal{L}\{x''(t)\} + \mathcal{L}\{x(t)\} = \mathcal{L}\{t\} \quad .$$

Using equation (5), and equation (8), we obtain

$$s^2X(s) - sx(0) - x'(0) - X(s) = \frac{1}{s^2} \quad .$$

Using the initial values, $x(0) = 3$ and $x'(0) = 0$, and simplifying gives

$$X(s) = \frac{1+3s^3}{s^2\big(s^2-1\big)} = \frac{2}{s-1} + \frac{1}{s+1} - \frac{1}{s^2} \quad .$$

From the table of Laplace transforms, we find the inverse Laplace transform of X(s) is

$$x(t) = 2e^t + e^{-t} - t \quad .$$

But $x(t) = y(t+2)$, therefore $y(t) = x(t-2)$. Hence we have

$$y(t) = 2e^{(t-2)} + e^{(2-t)} + 2 - t \quad .$$

For more examples, I highly recommend [7].

6.1 System Of ODEs

The Laplace transform can be used to solve a system of ordinary differential equations.

Example 1: Find the solution to the initial value problem

$$\begin{cases} x' = y + \sin(t) & x(0) = 2 \\ y' = x + 2\cos(t) & y(0) = 0 \end{cases} \quad .$$

We begin by taking Laplace transform of both sides, of both the equations, and using the initial conditions to obtain

$$\begin{cases} sX(s) - 2 = Y(s) + \frac{1}{s^2+1} \\ sY(s) = X(s) + \frac{2s}{s^2+1} \end{cases}$$

knowing the Laplace transform of $\sin(t)$ from section [2], the Laplace transform of $\cos(t)$ from section [3] and using equation (4). We proceed by eliminating either $X(s)$ or $Y(s)$. Eliminating $X(s)$, gives

$$s^2 - 1 - Y(s) = \frac{2s^2}{s^2+1} + 2 + \frac{1}{s^2+1} \quad .$$

Thus

$$Y(s) = \frac{4s^2+3}{(s^2+1)(s^2-1} = \frac{1}{2(s^2+1)} - \frac{7}{4(s+1)} + \frac{7}{4(s-1)} \quad .$$

Using equation (10), gives

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \quad .$$

Hence

$$y(t) = \frac{1}{2}\sin(t) - \frac{7}{4}e^{-t} + \frac{7}{4}e^t \quad .$$

We find $x(t)$ simply:

$$y' = x + 2\cos(t) \quad , \quad y' = \frac{1}{2}cos(t) + \frac{7}{4}e^{-t} + \frac{7}{4}e^t \quad .$$

Therefore, we obtain

$$x(t) = \frac{7}{4}e^{-t} + \frac{7}{4}e^t - \frac{3}{2}\cos(t) \quad .$$

6.2 Impulse-Response Function

Definition: The **transfer function** $R(s)$ is defined as the ratio of the Laplace transform of the output $x(t)$ to the Laplace transform of the input $x(t)$, as [2, 3, 7] suggest, given that the initial conditions are zero. This is equivalent to

$$R(s) = \frac{Y(s)}{X(s)} \quad .$$

Consider a second order differential equation

$$ay'' + by' + cy = x(t)$$

for a, b, c are constants with $y(0) = 0$ and $y'(0) = 0$. Taking the Laplace transform of both sides, using equation (3), and equation (5), gives

$$(as^2 + bs + c)Y(s) - asy(0) - ay'(0) - by(0) = X(s)$$

and using the fact that the initial conditions are zero, we have

$$(as^2 + bs + c)Y(s) = X(s) \quad .$$

Therefore, we define the transfer function to be

$$R(s) = \frac{Y(s)}{X(s)} = \frac{1}{as^2 + bs + c} \quad .$$

The function $r(s)$ is the **impulse-response function** defined as

$$\mathcal{L}^{-1}\{R(s)\} \quad .$$

Example: Using the convolution theorem obtain the solution to the following initial value problem

$$y'' - 2y' + y = x(t)$$

given $y(0) = -1$ and $y'(0) = 1$. We know the form of the transfer function and in our case we have

$$R(s) = \frac{1}{as^2 + bs + c} = \frac{1}{s^2 - 2s + 1} = \frac{1}{(s-1)^2} \quad .$$

From the table of Laplace transforms, see [3, 5, 7], the inverse of the Laplace transform of $R(s)$ is

$$r(s) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^t \quad .$$

We now solve the homogeneous problem, $y'' - 2y' + y = 0$, using the initial conditions $y(0) = -1$ and $y'(0) = 1$ to obtain

$$y(t) = (2t - 1)e^t \quad .$$

Hence, using the convolution theorem, equation (11), the solution to the initial value problem is

$$(x * r)(t) + y(t) = \int_0^t x(\tau)e^{t-\tau}(t - \tau)d\tau + (2t - 1)e^t$$

where r is the impulse-response function and $y(t)$ is the unique solution.

7 Dirac Delta Functional

The Dirac delta functional $\delta(t)$ is defined by

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0. \end{cases}$$

with the property:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad .$$

Various forms of the definition can be found, see [3, 4]. The shifted $\delta(t)$ is defined by

$$\delta(t-a) = \begin{cases} 0, & t \neq a \\ \int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a), & t = a. \end{cases}$$

The Laplace transform of $\delta(t - a)$, where $f(t) = e^{-st}$, is computed as

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\delta(t-a)dt = \int_{-\infty}^{\infty} e^{-st}\delta(t-a)dt = e^{-as} \quad .$$

We can find a relation, as highlighted in [7], between the $\delta(t)$ and the unit step function, $U(t)$ (see appendix A):

$$\int_{-\infty}^{t} \delta(x-a)dx = U(t-a) \quad .$$

Thus

$$\delta(t-a) = U'(t-a) \quad ,$$

that is the derivative of the Dirac delta function is the unit step function.

Example: Boundary-value problem A beam of 2λ that is embedded in a support on the left and free on the right. The vertical deflection of the beam a distance x away from the support is denoted by $y(x)$. If the beam has a concentrated load L on it in the center of the beam then the deflection must satisfy the boundary value problem

$$\begin{cases} EIy''''(x) = L\delta(x-\lambda) \\ y(0) = y'(0) = y''(2\lambda) = y'''(2\lambda) = 0 \end{cases}$$

where E is the modulus of elasticity and I is the moment of inertia, are constants. We shall solve the formula for the displacement $y(x)$ in terms of constants λ, L, E, and I.

We begin by taking the Laplace transform of both sides to obtain

$$EI\mathcal{L}\{y''''(x)\} = L\mathcal{L}\{\delta(x-\lambda)\} \quad .$$

By equation (5), and the initial conditions $y(0) = y'(0) = 0$

$$\mathcal{L}\{y''''(x)\} = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4Y(s) - sy''(0) - y'''(0) \quad .$$

Thus

$$s^4Y(s) - sy''(0) - y'''(0) = \frac{L}{EI}\mathcal{L}\{\delta(x-\lambda)\} = \frac{L}{EI}e^{-\lambda(s)} \quad .$$

Let $A = y''(0)$ and $B = y'''(0)$, then

$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} + \frac{L}{EI}\frac{e^{-\lambda(s)}}{s^4} \quad .$$

We now need to use equation (8), and equation (13), to find the inverse Laplace transform $Y(s)$

$$y(x) = \frac{Ax^2}{2} + \frac{Bx^3}{6} + \frac{L}{6EI}(x-\lambda)^3U(x-\lambda) = \frac{1}{6}\left\{\frac{6Ax^2}{2} + Bx^3 + \frac{L}{EI}(x-\lambda)^3U(x-\lambda)\right\} \quad .$$

We are given that $y''(2\lambda) = y'''(2\lambda) = 0$, then differentiating twice and thrice respectively, we obtain

$$0 = 6A + 12\lambda B + 6\lambda\frac{L}{EI}$$

and

$$0 = 6B + 6\frac{L}{EI} \quad .$$

Hence, the solution to the problem is

$$y(x) = \frac{L}{6EI}\{3\lambda x^2 - x^3 + (x-\lambda)^3U(x-\lambda)\} \quad .$$

For further referencing on the application of the Dirac delta function, I would suggest [3, 4, 7].

8 Partial Differential Equations

In this section, we show how to use the Laplace transform to solve one-dimensional linear partial differential equations. Partial differential equations are also known as PDEs.

There are 3 main steps in order to solve a PDE using the Laplace transform:

1. Begin by taking the Laplace transform with one of the two variables, usually t. This will give an ODE of the transform of the unknown function.

2. Solving the ODE, we shall obtain the transform of the unknown function.

3. By taking the inverse Laplace transform, we obtain the solution to the original problem.

8.1 The Heat Equation

In this section, through the use of the Laplace transforms, we seek solutions to initial-boundary value problems involving the heat equation.

The one-dimensional partial differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad . \tag{12}$$

is known as the **heat equation**, where c^2 is known as the thermal diffusivity of the material.

Example 1: Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \quad x > 0 \quad , \quad t > 0$$

given

$$u(x,0) = 0 \quad , \quad u(0,t) = \delta(t) \quad , \quad \lim_{x\to\infty} u(x,t) = 0 \quad .$$

We have to solve the heat equation for positive x and t, with $c^2 = 1$, subject to the boundary conditions

$$u(0,t) = \delta(t) \quad , \quad \lim_{x\to\infty} u(x,t) = 0 \quad ,$$

with the initial condition

$$u(x,0) = 0 \quad .$$

We begin by taking the Laplace transform, with respect to t, of both sides

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\mathcal{L}\{u(x,t)\} = \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} \quad .$$

Let $\mathcal{L}\{u(x,t)\} = U(x,s)$, then

$$sU = \frac{\partial^2 U}{\partial x^2} \Longrightarrow \frac{\partial^2 U}{\partial x^2} - sU = 0 \quad .$$

Notice that we have obtained an ODE, in the variable U, which has a general solution

$$U(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} \quad .$$

Applying the boundary conditions, with $\mathcal{L}\{f(t)\} = F(s)$, we obtain

$$U(0,s) = \mathcal{L}\{u(0,t)\} = \mathcal{L}\{\delta(t)\}) = 1 \quad ,$$

and

$$\lim_{x\to\infty} U(x,s) = \lim_{x\to\infty} \int_0^\infty e^{-st} u(x,t)dt = \int_0^\infty e^{-st} \lim_{x\to\infty} u(x,t)dt = 0 \quad .$$

The boundary condition

$$\lim_{x\to\infty} U(x,s) \Rightarrow A(s) = 0 \quad ,$$

as for every fixed $s > 0$, $e^{\sqrt{s}x}$ increases as $x \to \infty$. Hence

$$U(0,s) = B(s) = 1 \quad .$$

Therefore

$$U(x,s) = e^{-\sqrt{s}x} \quad .$$

From the tables of the Laplace transforms, see [2, 4], we obtain the inverse Laplace transform as

$$e^{-\sqrt{s}x} = \frac{x}{2\sqrt{\pi t^3}} e^{-\frac{x^2}{4t}} \quad .$$

Hence

$$u(x,t) = \frac{x}{2\sqrt{\pi t^3}} e^{-\frac{x^2}{4t}} \quad .$$

Example 2: We find the temperature $w(x,t)$ in a semi-infinite laterally insulated bar extending from $x = 0$ along the x-axis to infinity, assuming that the original temperature is 0, $w(x,t) \to 0$ as $x \to \infty$ for every fixed $t \geq 0$ and $w(0,t) = \frac{1}{\sqrt{t}}$.

We have to solve the heat equation for positive x and t subject to the boundary conditions

$$w(0,t) = \frac{1}{\sqrt{t}} \qquad , \qquad \lim_{x\to\infty} w(x,t) = 0$$

with the initial condition

$$w(x,0) = 0 \quad .$$

We begin by taking the Laplace transform, with respect to t, of both sides

$$\mathcal{L}\left\{\frac{\partial w}{\partial t}\right\} = s\mathcal{L}\{w(x,t)\} - 0 = c^2\mathcal{L}\left\{\frac{\partial^2 w}{\partial x^2}\right\} \quad .$$

Let $\mathcal{L}\{w(x,t)\} = W(x,s)$, then

$$sW = c^2\frac{\partial^2 W}{\partial x^2} \Longrightarrow \frac{\partial^2 W}{\partial x^2} - \frac{s}{c^2}W = 0 \quad .$$

Notice that we have obtained an ODE, in the variable W, which has a general solution

$$W(x,s) = A(s)e^{\frac{\sqrt{s}x}{c}} + B(s)e^{-\frac{\sqrt{s}x}{c}} \quad .$$

Applying the first boundary condition, we have

$$W(0,s) = \mathcal{L}\{w(0,t)\} = \sqrt{\frac{\pi}{s}} \Longrightarrow W(0,s) = A(s) + B(s) = \sqrt{\frac{\pi}{s}} \quad .$$

Assuming we can interchange the limit, the second boundary condition gives

$$\lim_{x\to\infty} W(x,s) = \lim_{x\to\infty}\int_0^\infty e^{-st}w(x,t)dt = \int_0^\infty e^{-st}\lim_{x\to\infty} w(x,t)dt = 0 \quad ,$$

thus $A(s) = 0$ since $c > 0$, and for every fixed $s > 0$, $e^{\frac{\sqrt{s}x}{c}}$ increases as $x \to \infty$. Hence

$$W(0,s) = B(s) = \sqrt{\frac{\pi}{s}} \quad .$$

Therefore

$$W(x,s) = \sqrt{\frac{\pi}{s}}e^{-\frac{\sqrt{s}x}{c}} \quad .$$

From the tables of the Laplace transforms, we obtain the inverse

$$\frac{e^{-\sqrt{s}\frac{x}{c}}}{\sqrt{s}} = \frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4c^2t}} \quad .$$

Hence

$$w(x,t) = \frac{1}{\sqrt{t}}e^{-\frac{x^2}{4c^2t}} \quad .$$

By the convolution theorem, see section [5], we can also express the solution as

$$w(x,t) = \int_0^t e^{-\frac{x^2}{4c^2\tau}}(t-\tau)^{-\frac{1}{2}}d\tau \quad .$$

9 Conclusion

In this study, we covered the basic properties of the Laplace transform and looked at some applications. We, through the use of examples, illustrated how the properties of the Laplace transform can be used in order to simplify, and solve problems. In particular, we looked at solving ordinary and partial differential equations through the use of the Laplace transforms.

We can extend our study of the Laplace transforms to include differential equations that govern current, charge, capacitance and resistance within electrical circuits, see [4]. We can also extend our study of the Laplace transforms to cover the Z-transform, the discrete counterpart of the Laplace transform. With the use of the Z-transforms we can include examples of solutions to difference equations. I would suggest [6], as a guide to the Z-transforms.

A Unit Step Function

The unit step function, $U(t)$, is defined by

$$U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

For $t \geq 0$ the unit step function is the same as 1. Therefore, as [4] suggests, the Laplace transform of $U(t)$ is

$$\mathcal{L}\{U(t)\} = \mathcal{L}\{1\} = \frac{1}{s} \quad .$$

Define the shifted unit step function, $U(t-a)$ as

$$U(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a, \end{cases}$$

then the Laplace transform of $U(t-a)$ is

$$\mathcal{L}U(t-a) = \int_0^\infty U(t-a)e^{-st}dt = \int_a^\infty e^{-st}dt = \frac{e^{-as}}{s} \quad .$$

A.1 Second Shift Theorem

Theorem 3. *Let $x(t)$ be a function, then*

$$\mathcal{L}\{x(t-a)U(t-a)\} = e^{-as}X(s) \quad . \tag{13}$$

It should be clear that

$$\mathcal{L}^{-1}\{e^{-as}X(s)\} = x(t-a)U(t-a) \quad .$$

For the proof of the "Second Shift Theorem", see [4].

References

[1] A.D. Osborne, *Complex Variables And Their Applications,* Addison Wesley Longman, 1998.

[2] E. Kreyszig, *Advanced Engineering Mathematics,* John Wiley & Sons, 2005.

[3] A. Pinkus, S. Zafrany, *Fourier Series And Integral Transforms,* Cambridge University Press, 1997.

[4] P.P.G. Dyke, *An Introduction to Laplace Transforms And Fourier Series,* Springer, 2000.

[5] J.D. Logan, *A First Course in Differential Equations,* Springer, 2005.

[6] W. Bolton, *Laplace and Z-transforms (Mathematics For Engineers),* Longman, 1993.

[7] R.K. Nagle, E.B. Saff *Fundamentals of Differential Equations,* Addison Wesley, 1996.